Subgroups of Diagram Group Using Covering Methods

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Abstract

Diagram groups are created from geometric objects named semigroup presentation S. These diagrams are drawn and considered as 2-complexes. The aim of this article is to determine the covering complex for diagram groups over union of two semigroup presentations by adding some initial generators and relations to alphabet and a set of relation correspondingly. The main aim is to study subgroups of diagram group, we present a method of producing normal subgroups of one generator. In addition, we design a new method for computing all generators and relations for the fundamental group $\pi_1(K({}^3S_1 \cup {}^3S_2, W)$.

Keywords: Diagram groups, Semigroup presentation, Generators, Relation.

1. Introduction

In [7] we obtained the connected 2-complex space ${}^{3}K_i$ that were obtained from ${}^{3}S =< a,b,c: a = b, b = c, c = a >$, also we prove that the connected 2-complex graph ${}^{3}K_{i+1}$ is the covering complex for ${}^{3}K_i$ for all $i \geq 2$.

In this paper we want to determine the semigroup presentation of union of two semigroup presentations by adding a relation.

Let ${}^{3}S_1 =< a,b,c: a = b, b = c, c = a >$, ${}^{3}S_2 =< x,y,z: x = y, y = z, z = x >$ and ${}^{3}S =< a,b,c,x,y,z: a = b, b = c, c = a, x = y, y = z, z = x, x = a >$ be semigroup presentations. Now, we consider the semigroup presentation ${}^{3}S$ obtained from union of initial generators and relations ${}^{3}S_1$ and ${}^{3}S_2$ by adding a relation $x = a$ (for more details see [3], and [5]).

In fact, Guba and Sapir (1997) have shown that if $S_1 =< X_1: r_1 >$, $S_2 =< X_2: r_2 >$ and $S =< X_1 \cup X_2: r_1 \cup r_2 >$are semigroup presentations, then for $W_1, W_2 \in D(S, W_1W_2)$is isomorphic to the direct product of$D(S, W_1)$ and $D(S, W_2)$. Also, they proved if one considers$S =< X_1 \cup X_2: r_1 \cup r_2 \cup \{W_1 = W_2\} >$ where $X_1, X_2$are disjoint sets, and the congruence class ofW_1modulesS_1does not contain words of the form YW_iZ, where Y, Zare words over$X_1, X_2$and YZ are not empty. Then $D(S, W_i)$is isomorphic to the free product$D(S_1, W_i)$and$D(S_2, W_i)$.

In section 2, we will determine the connected 2-complex space ${}^{3}K_i, i \in N$obtained from semigroup presentation ${}^{3}S =< a,b,c,x,y,z: a = b, b = c, c = a, x = y, y = z, z = x, a = x >$.

In section 3, we will compute the generators and relations for the fundamental group $\pi_1(K({}^{3}S_1 \cup {}^{3}S_2, W)$.

2. Determining the connected 2-complex graphs

Let ${}^{3}S =< a, b, c, x, y, z: a = b, b = c, c = a, x = y, y = z, z = x, a = x >$ be a semigroup presentation. Associated with semigroup presentation $S =< X: r >$ we have a connected 2-complex graph ${}^{3}K_{i}, i \in N$ where the vertices are word on set of alphabet X and the edges are the form $e = (W_1, R_\varepsilon \to R_{-\varepsilon}, W_2)$such that$\iota(e) = W_1 R_\varepsilon W_2$, $\tau(e) = W_1 R_{-\varepsilon} W_2$.

The connected 2-complex obtained from$S$is collections of subgraphs${}^{3}K_{i}, i \in N$. Note that the 2-complexgraph${}^{3}K_{i}({}^{3}S_{1})$obtained from semigroup presentation ${}^{3}S_{1}$ is just collection of subgraphs ${}^{3}K_{i}({}^{3}S_{1})$where ${}^{3}K_{i}({}^{3}S_{1})$ contains all vertices of length n and respective edges. Similar we obtain ${}^{3}K_{i}({}^{3}S_{2})$ from semigroup presentation ${}^{3}S_{2}$for the graph ${}^{3}K_{i}({}^{3}S_{2})$. Now for semigroup presentation ${}^{3}S$, the graph ${}^{3}K_{i}({}^{3}S) = {}^{3}K_{i}({}^{3}S_{1}) \cup {}^{3}K_{n}({}^{3}S_{2}) \cup \{(u, a \to x, v\}$ such that the length $uv = n - 1$. IfW_n is a vertex in ${}^{3}K_{n}({}^{3}S)$then $W_n h$, $h \in \{a, b, c, x, y, z\}$is a vertex in ${}^{3}K_{n+1}({}^{3}S)$. Similarly, if $(u, R_\varepsilon \to R_{-\varepsilon}, v)$ is edge in ${}^{3}K_{n}({}^{3}S)$, then $(u, R_\varepsilon \to R_{-\varepsilon}, vh)$is the respective edge in ${}^{3}K_{i+1}({}^{3}S)$. Thus ${}^{3}K_{i+1}({}^{3}S)$ is just six copies of ${}^{3}K_{i}({}^{3}S)$togetherwithsixverticesandedges$(u, R_\varepsilon \to R_{-\varepsilon}, vh)$, $h \in \{a, b, c, x, y, z\}$.

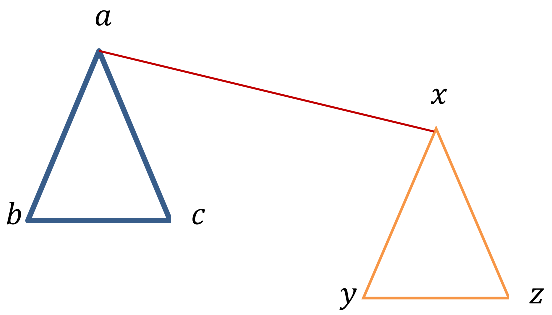

Figure 1: The connected 2-complex graph${}^{3}K_{1}({}^{3}S)$

While ${}^{3}K_2({}^{3}S)$ is

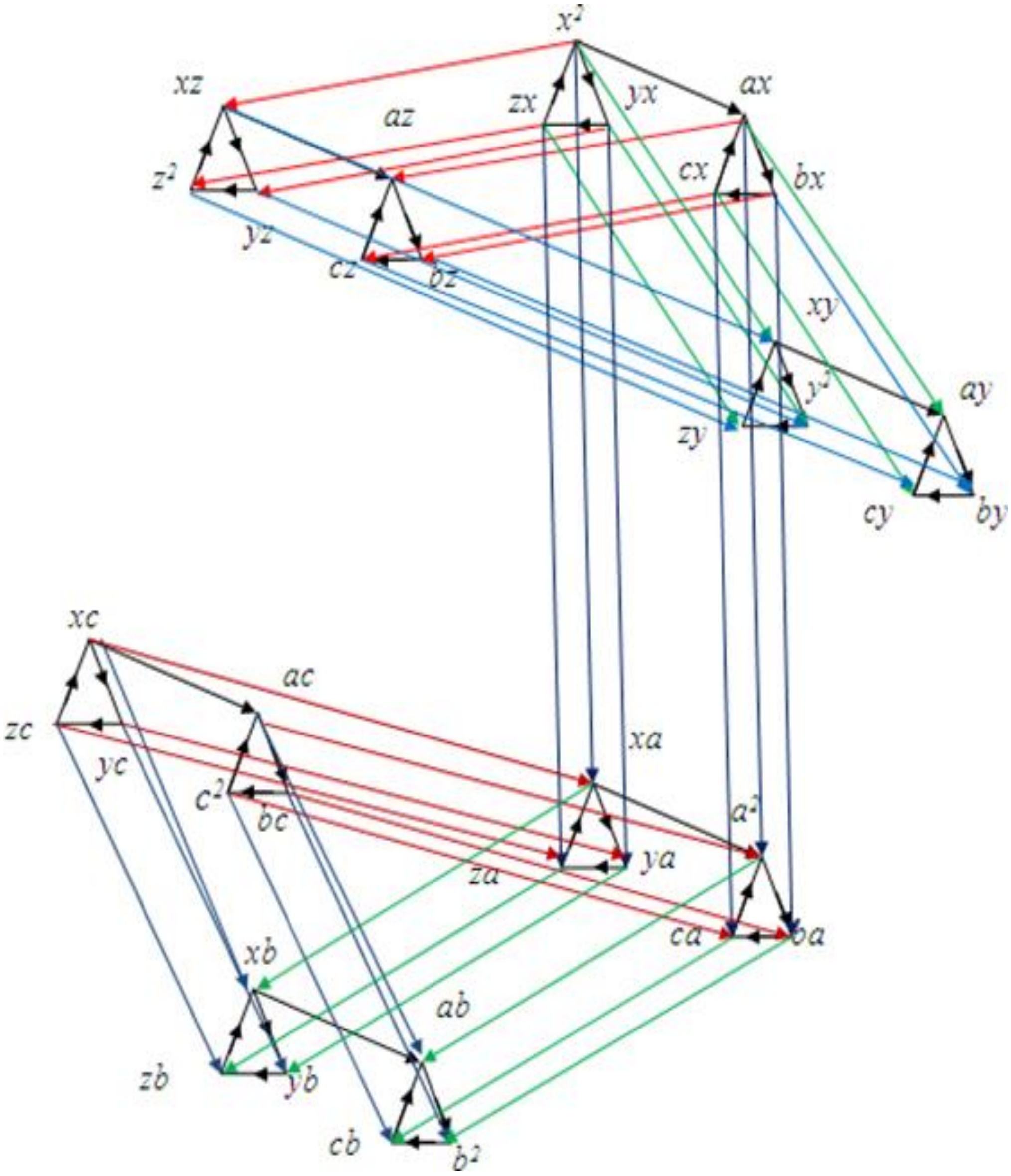

Figure 2: The connected 2-complex graph ${}^{3}\boldsymbol{K_2}({}^{3}\boldsymbol{S})$

Note that ${}^3K_2({}^3S)$is six copies of ${}^3K_1({}^3S)$and each vertex in each copy are joinedtogether respectively. Likewise with six copies of ${}^3K_3({}^3S)$, the 2-complex graph ${}^3K_2({}^3S)$may be obtained by repeating similarprocedureswithresult ${}^3K_4({}^3S)$and so on.

Observation2.1 A connected 2—complex graph${}^3K_i({}^3S)$contains 6^i vertices.

Observation2.2 A connected 2 -complex graph ${}^3K_{n+1}({}^3S)$is six copies of ${}^3K_i({}^3S)$.Thus,if there is $e_{l,m}$edges in ${}^3K_i({}^3S)$then the number of edges in ${}^3K_{i+1}({}^3S)$ is $6e_{l,m}$plus all edges between triangles in${}^3K_{i+1}({}^3S)$,which is $e = 6e_{i-1} + 6^i + 6^{i-1}$.

Observation2.3 Vertices$U$and$V$are connected if and only if $L(U) = L(V)$.

Lemma2.4 If$L(U) = L(V)$then$\pi_1({}^3K_i({}^3S),U) = \pi_1({}^3K_i({}^3S),V)$.

Lemma2.5 Vertices of ${}^3K_i({}^3S)$are all words of length i.

3. Subgroups of Diagram Group Using Covering Methods

Let W be a positive word on ${}^3S$. When the length of W equal to one, then we have 6^1possibilities vertices in the connected 2-complex graph ${}^3K_1({}^3S)$namely x, y, z, a, b, c as is shown Figure 1.

Definition 3.1

A 2-complex graph ${}^3K_H$ contains the following:

i. Vertices: The set of right cosets $H[\alpha]$of H, where $[\alpha] \in P_v, v \in V$.

ii. Edges: All ordered pairs $(H[\alpha], x^{\varepsilon})$where x is an edge in K and $\varepsilon = \pm 1$.

iii. Functions:

a. $i(H[\alpha], x^{\varepsilon}) = H[\alpha]$.

b. $\tau(H[\alpha], x^{\varepsilon}) = H[\alpha x^{\varepsilon}]$.

c. $(H[\alpha], x^{\varepsilon})^{-1} = (H[\alpha x^{\varepsilon}], x^{-\varepsilon})$.

Theorem 3.2 (Rotman 1995,2002)

The component ${}^{3}K_H$ is connected 2-complex graph.

Theorem 3.3 (Rotman 1995,2002)

The map φ_H:${}^{3}K_H({}^{3}S) \rightarrow {}^{3}K({}^{3}S)$, $\varphi_H(H[\alpha]) = v$, $\varphi_H(H[\alpha], x) = x$ is a mapping of connected 2-complex graphs.

Theorem 3.4 (Rotman 1995,2002)

The map φ_H:${}^{3}K_H({}^{3}S) \rightarrow {}^{3}K({}^{3}S)$, $\varphi_H(H[\alpha]) = v$, $\varphi_H(H[\alpha], x) = x$isa locally bijective.

Theorem 3.5 (Rotman 1995,2002)

The mapping $\varphi_H^*: \pi_1({}^{3}K_H({}^{3}S), v') \rightarrow \pi_1({}^{3}K({}^{3}S), v)$ is an injective if φ_H^*isa locally bijective.

Theorem 3.6

Consider the following connected 2-complex graph${}^{3}K_1({}^{3}S)$ as shown in Figure 1, such that $G = \pi_1({}^{3}K_1({}^{3}S), a)$ contains δ_1, where $\delta_1 =< e_{a,b}e_{b,c}e_{c,a} >$. If $H_{1_2}$is the smallest normal subgroup of Gcontaining $< \delta_1^2 >$, then the covering complex ${}^{3}K_{H_{1_2}}({}^{3}S)$ for${}^{3}K_1({}^{3}S)$is two hexagons' shapes.

Proof: Use the notion of $H[\gamma] = H[\beta] \Leftrightarrow [\gamma\beta^{-1}] \in H$. From ${}^{3}K_1({}^{3}S)$, $\pi_1({}^{3}K_1({}^{3}S))$ can be obtained. Fix a vertex a in ${}^{3}K_1({}^{3}S)$. Now, for any normal subgroup of$\pi_1({}^{3}K_1({}^{3}S), a)$, there exists a unique covering space, start by choosing basic $H[\alpha]$ where α is a path such that $i(\alpha) = a, \tau(\alpha) = v$ for every vertex $v$in ${}^{3}K_1({}^{3}S)$. As a result, these basic $H[1], H[e_{a,b}]$, and $H[e_{a,b}e_{b,c}]$ will be selected, and then all possible edges, vertices, can be determined as shown in Table 1 and Table 2.

Table 1: Vertices in ${}^{3}K_1({}^{3}S)$ and ${}^{3}K_{H_{1_2}}({}^{3}S)$

Vertices in ${}^{3}K_1({}^{3}S)$	Vertices in ${}^{3}K_{H_{1_2}}({}^{3}S)$
a	$H[1]$
b	$H[e_{a,b}]$
c	$H[e_{a,b}e_{b,c}]$
a	$H[e_{a,b}e_{b,c}e_{c,a}]$
b	$H[e_{a,b}e_{b,c}e_{c,a}e_{a,b}]$
c	$H[e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c}]$
x	$H[e_{a,x}]$
y	$H[e_{a,x}e_{x,y}]$
z	$H[e_{a,x}e_{x,y}e_{y,z}]$
x	$H[e_{a,x}e_{x,y}e_{y,z}e_{z,x}]$
y	$H[e_{a,x}e_{x,y}e_{y,z}e_{z,x}e_{x,y}]$
z	$H[e_{a,x}e_{x,y}e_{y,z}e_{z,x}e_{x,y}e_{y,z}]$

Table 2: Edges in $^3K_1(^3S)$ and $^3K_{H_{1_2}}(^3S)$

Edges in $^3K_1(^3S)$	Edges in $^3K_{H_{1_2}}(^3S)$
$e_{a,b}$	$(H[1], e_{a,b})$
$e_{a,b}e_{b,c}$	$(H[e_{a,b}], e_{a,b}e_{b,c})$
$e_{a,b}e_{b,c}e_{c,a}$	$(H[e_{a,b}e_{b,c}], e_{a,b}e_{b,c}e_{c,a})$
$e_{a,b}$	$(H[e_{a,b}e_{b,c}e_{c,a}], e_{a,b}e_{b,c}e_{c,a}e_{a,b})$
$e_{a,b}e_{b,c}$	$(H[e_{a,b}e_{b,c}e_{c,a}e_{a,b}], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})$
$e_{a,b}e_{b,c}e_{c,a}$	$(H[1], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})$
$e_{a,x}$	$(H[1], e_{a,x})$
$e_{a,x}e_{x,y}$	$(H[e_{a,x}], e_{a,x}e_{x,y})$
$e_{a,x}e_{x,y}e_{y,z}$	$(H[e_{a,x}e_{x,y}], e_{a,x}e_{x,y}e_{y,z})$
$e_{a,x}e_{x,y}e_{y,z}e_{z,x}$	$(H[e_{a,x}e_{x,y}e_{y,z}], e_{a,x}e_{x,y}e_{y,z}e_{z,x})$
$e_{a,x}$	$(H[e_{a,x}e_{x,y}e_{y,z}e_{z,x})], e_{a,x}e_{x,y}e_{y,z}e_{z,x}e_{a,x})$
$e_{a,x}e_{x,y}$	$(H[e_{a,x}e_{x,y}e_{y,z}e_{z,x}e_{a,x})], e_{a,x}e_{x,y}e_{y,z}e_{z,x}e_{a,x}e_{x}$
$e_{a,x}e_{x,y}e_{y,z}$	$(H[1], e_{a,x}e_{x,y}e_{y,z}e_{z,x}e_{a,x}e_{x,y}e_{y,z})$

Now, suppose $\varphi_H: {}^3K_{H_{1_2}}(^3S) \rightarrow {}^3K_1(^3S)$defined by $\varphi_H(H[1]) = a, \varphi_H[e_{a,b}] = b$,

$\varphi_H(H[1], e_{a,b}) = e_{a,b}$. This map can be viewed as a locally bijective. For this reason, $^3K_{H_{1_2}}(^3S)$is a covering graph for$^3K_1(^3S)$and it is of two hexagons shapes. Therefore, the covering space $^3K_{H_{1_2}}(^3S)$ for $^3K_1(^3S)$ in this case is of two hexagons shapes.

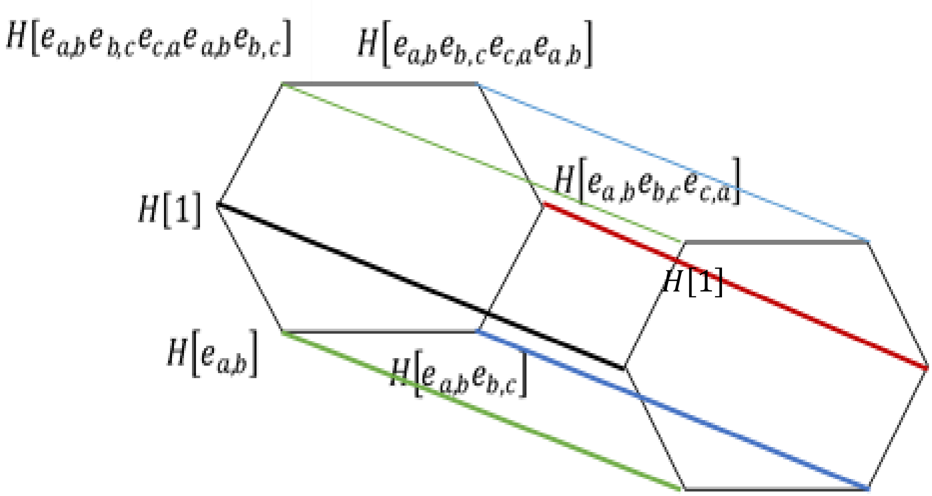

Figure 3: The connected 2-complex graph $^3K_{H_{1_2}}(^3S)$

Since a is a vertex of the connected 2-complex graph $^3K_2(^3S)$, and $H[1]$ lies over a, then by theorem 3.5 $\varphi_H^*: (\pi_1(^3K_{H_{1_2}}(^3S), H[1])) \to Im\varphi_H^* = H$. As a result, $H = \pi_1(^3K_{H_{1_2}}(^3S), H[1])$ can be obtained as a subgroup of $G = \pi_1(^3K_1(^3S), a)$.

The generators for $\pi_1(^3K_{H_{1_2}}(^3S), H[1])$ are computed here using maximal subtree methods, select $T(^3K_{H_{1_2}}(^3S))$ for $^3K_{H_{1_2}}(^3S)$ (see Figure 4).

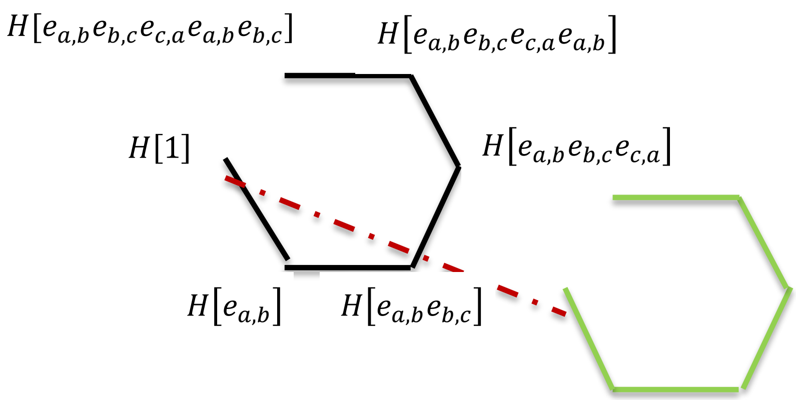

Figure 4: The maximal tree $T(^3K_{H_{1_2}}(^3S))$

The generators for the fundamental group $\pi_1(^3K_{H_{1_2}}(^3S), H[1])$ will be:

1. $g_1(^3K_{H_{1_2}}(^3S)) =$
   $(H[1], e_{a,b})(H[e_{a,b}], e_{a,b}e_{b,c})(H[e_{a,b}e_{b,c}], e_{a,b}e_{b,c}e_{c,a})$

2. $(H[e_{a,b}e_{b,c}e_{c,a}], e_{a,b}e_{b,c}e_{c,a}e_{a,b})$

$(H[e_{a,b}e_{b,c}e_{c,a}e_{a,b}], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})(H[1], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})^{-1}$.

3. $g_2(^3K_{H_{1_2}}(^3S)) = (H[1], e_{a,b})(H[1], e_{a,x})(H[e_{a,x}], e_{a,x}e_{x,y})$

$(H[e_{a,x}e_{x,y}], e_{a,x}e_{x,y}e_{y,z})(H[e_{a,x}e_{x,y}e_{y,z}], e_{a,x}e_{x,y}e_{y,z}e_{z,x})$

$(H[e_{a,x}e_{x,y}e_{y,z}e_{z,x})], e_{a,x}e_{x,y}e_{y,z}e_{z,x}e_{a,x})$

$(H[e_{a,x}e_{x,y}e_{y,z}e_{z,x}e_{a,x})], e_{a,x}e_{x,y}e_{y,z}e_{z,x}e_{a,x}e_{x,y})(H[1], e_{a,x}e_{x,y}e_{y,z}e_{z,x}e_{a,x}e_{x,y}e_{y,z})^{-1}$.

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